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Unitary Representations of Topological Groups and Functional Equations

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1. INTRODUCTION

In [6] T. A. O'Connor considered the functional equation

$$f(x-y)=\sum_{j=1}^{n} a_j(x)\,\overline{a_j(y)}, \tag{1}$$

where x and y run over a locally compact Abelian group G. All the functions f and a_j ($j=1, ..., n$) occurring in (1) are defined on G and assume values in the complex field $\mathbb{C}$. It was proved in [6] that for any system of continuous functions $f, a_1, ..., a_n$ satisfying Eq. (1), there exist continuous characters $\gamma_1, ..., \gamma_m$ of the group G (cf. [2, Chap. V, 22.15]) and positive numbes $\lambda_1, ..., \lambda_m$ such that f may be expressed in the form

$$f(x)=\sum_{i=1}^{m} \lambda_i \gamma_i(x), \qquad x \in G.$$

This result was obtained by means of the Bochner representation theorem for positive definite functions. The author of [6] confined himself to the aim of determining the form of the function f only. He was not concerned with the functions $a_1, ..., a_n$ and did not find necessary and sufficient conditions for them to satisfy Eq. (1). One of the aims of the present paper is to fill this gap.

To begin with, let us note that Eq. (1) may be briefly written as

$$f(x-y)=(a(x) \mid a(y)), \qquad x, y \in G, \tag{2}$$

where $a\colon G \to \mathbb{C}^n$ is given by the formula

$$a(x) := (a_1(x), ..., a_n(x)), \qquad x \in G$$

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and $(\cdot \mid \cdot)$ stands for the usual inner product in the space $\mathbb{C}^n$. Surprisingly enough, Eq. (2) can easily be solved even at a higher level of generality, where the function a is assumed to map G into an arbitrary Hilbert space $(H, (\cdot \mid \cdot))$. In this general setting, solutions of (2) can be described in terms of unitary representations of the group G (see Theorem 1). At this place we recall some relevant notions used in the sequel.

By a unitary representation of a topological group G in a Hilbert space H we mean a homomorphism of G into the group of all unitary operators in H (cf., e.g., [4, 5]).

A unitary representation U is said to be continuous iff for each vector $\xi \in H$ the transformation

$$G \ni x \to U(x)\, \xi \in H$$

is continuous.

A vector $\xi_0 \in H$ is called a cyclic vector of a representation U iff the space $\mathrm{cl}\,\mathrm{Lin}\{U(x)\, \xi_0 : x \in G\}$ coincides with the whole of H. Here and later on the symbol Lin A designates the linear space spanned by a set $A \subset H$, whereas cl stands for the closure operation in the norm topology of H.

Once we know all the solutions of Eq. (2) for an arbitrary Hilbert space H, we may reinterpret them in the special case where $H = \mathbb{C}^n$. In this way we determine the general form of solutions of Eq. (1), completing the previously mentioned result of O'Connor.

The second part of the present paper is devoted to the following functional equation of D'Alembert's type (cf., e.g., [1, 3, 7, 8]):

$$f(x+y) + f(x-y) = 2(a(x) \mid a(y)), \qquad x, y \in G. \tag{3}$$

Once more we assume that f is a complex-valued function defined on G, whereas a maps G into a Hilbert space $(H, (\cdot \mid \cdot))$. Under some additional assumptions on the function a, we describe the solutions of (3) in the language of unitary representations of the group G. Finally, in the same spirit as before, we draw respective conclusions for a finite-dimensional analogue of Eq. (3).

2. Solution of Equations (1) and (2)

Whenever the group G is not required to be Abelian, we apply the multiplicative notation (instead of additive) for the group operation in G. The identity element of G is then denoted by e. Moreover, we replace (2) by the functional equation

$$f(y^{-1}x) = (a(x) \mid a(y)), \qquad x, y \in G, \tag{4}$$

which coincides with (2) provided G is commutative.

THEOREM 1. *Let* $(G, \cdot)$ *be a topological group and let* $(H, (\cdot \mid \cdot))$ *be a Hilbert space. Suppose that* $f\colon G \to \mathbb{C}$ *and* $a\colon G \to H$ *are continuous functions. Then* f *and* a *satisfy Eq.* (4) *if and only if there exists a continuous unitary representation* U *of the group* G *in the space* $H_0 := \operatorname{cl}\operatorname{Lin} a(G)$ *with the cyclic vector* $\xi_0 := a(e)$ *such that*

$$a(x) = U(x)\,\xi_0, \qquad x \in G; \tag{5}$$

$$f(x) = (U(x)\,\xi_0 \mid \xi_0), \qquad x \in G. \tag{6}$$

Proof. Suppose that there exist a unitary representation U and a vector ξ_0 such that formulae (5) and (6) hold. Then we have

$$\begin{aligned} f(y^{-1}x) &= (U(y^{-1}x)\,\xi_0 \mid \xi_0) = (U(y)^{-1}U(x)\,\xi_0 \mid \xi_0) \\ &= (U(x)\,\xi_0 \mid U(y)\,\xi_0) = (a(x) \mid a(y)), \qquad x, y \in G, \end{aligned}$$

which means that f and a defined by (5) and (6) satisfy Eq. (4).

Conversely, assume that f and a are solutions of Eq. (4). Then, for all $x, w, z \in G$, we have

$$\begin{aligned} (a(xw) \mid a(xz)) &= f((xz)^{-1}xw) = f(z^{-1}x^{-1}xw) = f(z^{-1}w) \\ &= (a(w) \mid a(z)), \end{aligned}$$

whence,

$$(a(xw) \mid a(xz)) = (a(w) \mid a(z)), \qquad x, w, z \in G.$$

Now, fix a positive integer n, complex numbers $\alpha_1, \dots, \alpha_n$ and elements $x, w_1, \dots, w_n \in G$. On account of the preceding identity we get

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i a(xw_i) \right\|^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j (a(xw_i) \mid a(xw_j)) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j (a(w_i) \mid a(w_j)) \\ &= \left\| \sum_{i=1}^{n} \alpha_i a(w_i) \right\|^2, \end{aligned}$$

which yields

$$\left\| \sum_{i=1}^{n} \alpha_i a(xw_i) \right\| = \left\| \sum_{i=1}^{n} \alpha_i a(w_i) \right\|. \tag{7}$$

Next we put $\tilde{H} := \operatorname{Lin} a(G)$ and for each fixed $x \in G$, we define an operator $\tilde{U}(x)$ in $\tilde{H}$ by

$$\tilde{U}(x)\xi := \sum_{i=1}^{n} \alpha_i a(xw_i),$$

whenever $\xi = \sum_{i=1}^{n} \alpha_i a(w_i)$ for some $\alpha_i \in \mathbb{C}$, $w_i \in G$, $i = 1, \ldots, n$.

First observe that $\tilde{U}(x)$ is correctly defined. Indeed, if ξ admits two different expansions

$$\xi = \sum_{i=1}^{n} \alpha_i a(w_i) = \sum_{j=1}^{m} \beta_j a(z_j),$$

then (7) implies that

$$\left\| \sum_{i=1}^{n} \alpha_i a(xw_i) - \sum_{j=1}^{m} \beta_j a(xz_j) \right\| = \left\| \sum_{i=1}^{n} \alpha_i a(w_i) - \sum_{j=1}^{m} \beta_j a(z_j) \right\| = 0.$$

Evidently, for each $x \in G$, $\tilde{U}(x)$ is a linear operator in $\tilde{H}$ and (7) ensures that it is an isometry of $\tilde{H}$ into itself. It is also clear that $\tilde{U}(x)$ is surjective for every $x \in G$. Moreover, choosing elements $x, y \in G$ and an arbitrary vector $\xi \in \tilde{H}$ with the expansion

$$\xi = \sum_{i=1}^{n} \alpha_i a(w_i),$$

we can easily see that

$$\tilde{U}(xy)\xi = \sum_{i=1}^{n} \alpha_i a(xyw_i) = \tilde{U}(x)\, \tilde{U}(y)\xi,$$

whence,

$$\tilde{U}(xy) = \tilde{U}(x)\, \tilde{U}(y), \qquad x, y \in G.$$

Finally, for every $\xi = \sum_{i=1}^{n} \alpha_i a(w_i) \in \tilde{H}$, the transformation

$$G \ni x \to \sum_{i=1}^{n} \alpha_i a(xw_i) = \tilde{U}(x)\, \xi \in \tilde{H}$$

is continuous, since the function a is also.

Now, for each $x \in G$, let $U(x)$ denote the unique continuous extension of the operator $\tilde{U}(x)$ to the space $H_0 := \operatorname{cl} \tilde{H}$. Bearing in mind what has just been shown about the operators $\tilde{U}(x)$, $x \in G$, one can readily check that the transformation

$$G \ni x \to U(x)$$

derived in this way is a continuous unitary representation of the group G in the space H_0. It remains to show that the representation U has all the desired properties.

Setting $\xi_0 := a(e)$, we infer that $\xi_0 \in \tilde{H} \subset H_0$ and

$$U(x)\,\xi_0 = \tilde{U}(x)\xi_0 = a(xe) = a(x), \qquad x \in G,$$

which yields (5). To prove (6) we insert $y := e$ in Eq. (4). As a result we obtain

$$f(x) = (a(x) \mid a(e)) = (U(x)\xi_0 \mid \xi_0), \qquad x \in G.$$

The fact that ξ_0 is a cyclic vector of the representation U follows directly from (5) and from the definition of H_0, which guarantee that

$$\operatorname{cl}\operatorname{Lin}\{U(x)\xi_0 : x \in G\} = \operatorname{cl}\operatorname{Lin} a(G) = H_0.$$

Thus the proof is finished.

Remark 1. We assume nothing about the topology in the group G except that it converts G into a topological group. In particular, we may equip G with the discrete topology for which every function defined on G is continuous. In this case the continuity assumption occurring in Theorem 1 becomes redundant and the theorem allows us to solve Eq. (4) in the class of all functions (without any restrictions concerning their regularity). It is also worthwhile to note that the completeness of the space H is nowhere explicitly used in the proof of Theorem 1. In fact the only place where it intervenes is the definition of a unitary representation in which it is customary to assume that the representation space is complete. Apart from this requirement, H might have been an arbitrary inner product space.

Remark 2. From the form of Eq. (4) alone, it follows immediately that any function $f: G \to \mathbb{C}$ satisfying this equation (with a certain $a: G \to H$) must be positive definite (cf. [4, Chap. VI, Sect. 30]). Indeed, for each positive integer n, complex numbers $\alpha_1, ..., \alpha_n$, and elements $x_1, ..., x_n \in G$ we have

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j f(x_j^{-1} x_i) = \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (a(x_i) \mid a(x_j))$$
$$= \left\| \sum_{i=1}^{n} \alpha_i a(x_i) \right\|^2 \geqslant 0.$$

It is well known (see [4, Chap. VI, Sect. 30, Theorem 1]) that with the function f one may associate an abstract Hilbert space $(X, \langle \cdot \mid \cdot \rangle)$ and a unitary representation V of the group G in the space X with a cyclic vector $\eta_0 \in X$ such that

$$f(x) = \langle V(x)\eta_0 \mid \eta_0 \rangle, \qquad x \in G.$$

The essence of our theorem consists in the fact that f may actually be expressed by means of a unitary representation of G whose representation space is contained in the original Hilbert space H. Moreover, this representation is related to the function a by formula (5).

Now we return to the case of an Abelian group G and to a study of Eq. (1). By $M(m \times n; \mathbb{C})$ we denote the set of all complex matrices with m rows and n columns. We also use the Kronecker symbol δ^i_k defined by

$$\delta^i_k := \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k. \end{cases}$$

THEOREM 2. *Let $(G, +)$ be an Abelian topological group. Then continuous functions $f, a_j \colon G \to \mathbb{C}$ $(j = 1, ..., n)$ satisfy Eq. (1) if and only if there exist continuous characters $\gamma_1, ..., \gamma_m$ of the group G $(m \leqslant n)$, complex numbers $\alpha_1, ..., \alpha_m$, and a matrix $[\beta_{i,j}] \in M(m \times n; \mathbb{C})$ such that*

$$\sum_{j=1}^{n} \beta_{i,j}\bar{\beta}_{k,j} = \delta^i_k, \qquad i, k = 1, ..., m; \tag{8}$$

$$f(x) = \sum_{i=1}^{m} |\alpha_i|^2 \gamma_i(x), \qquad x \in G; \tag{9}$$

$$a_j(x) = \sum_{i=1}^{m} \alpha_i \beta_{i,j} \gamma_i(x), \qquad x \in G, j = 1, ..., n. \tag{10}$$

Proof. Suppose that f and a_j $(j = 1, ..., n)$ are continuous functions satisfying Eq. (1). If we choose the canonical orthonormal basis $(e_1, ..., e_n)$ of the n-dimensional space $\mathbb{C}^n$ and if we put

$$a(x) := \sum_{j=1}^{n} a_j(x)\, e_j, \qquad x \in G, \tag{11}$$

then Eq. (1) turns into

$$f(x - y) = (a(x) \mid a(y)), \qquad x, y \in G.$$

By virtue of Theorem 1, there exists a continuous unitary representation U of the group G in the space $H_0 := \mathrm{cl\,Lin}\, a(G) \subset \mathbb{C}^n$ such that

$$a(x) = U(x)\xi_0, \qquad x \in G;$$
$$f(x) = (U(x)\,\xi_0 \mid \xi_0), \qquad x \in G,$$

where $\xi_0 := a(0)$. The representation U, being finite-dimensional, is completely reducible (see [5, Chap. I, 2.8.IX]). The last statement means that the space H_0 admits a decomposition into the direct sum of mutually orthogonal non-zero subspaces $H_1, ..., H_m$ which are invariant under U and do not contain proper invariant subspaces. Moreover, to each subspace H_i $(i = 1, ..., m)$ there corresponds a continuous irreducible unitary representation U_i of the group G in the space H_i such that

$$U(x)\xi = U_1(x)\xi_1 + \cdots + U_m(x)\xi_m \tag{12}$$

for $x \in G$ and $\xi = \xi_1 + \cdots + \xi_m$, $\xi_i \in H_i$, $i = 1, ..., m$. Since G is Abelian, the irreducible representations U_i are in fact one-dimensional, i.e., $\dim H_i = 1$ for $i = 1, ..., m$ (see [5, Chap. I, 2.2, Corollary]). Therefore one can find continuous characters $\gamma_1, ..., \gamma_m$ of the group G such that

$$U_i(x)\xi_i = \gamma_i(x)\xi_i \qquad \text{for} \quad x \in G \text{ and } \xi_i \in H_i,\ i = 1, ..., m \tag{13}$$

(see [2, Chap. V, 22.16b]).

Now, for each $i = 1, ..., m$, select a vector $v_i \in H_i$ with $\|v_i\| = 1$. Then the vectors $v_1, ..., v_m$ form an orthonormal basis of the space H_0. Let

$$\xi_0 = a(0) = \sum_{i=1}^{m} \alpha_i v_i$$

for some $\alpha_1, ..., \alpha_m \in \mathbb{C}$. Then on account of (12) and (13) we have

$$U(x)\xi_0 = \sum_{i=1}^{m} \gamma_i(x)\,\alpha_i v_i, \qquad x \in G$$

and consequently,

$$\begin{aligned} f(x) = (U(x)\xi_0 \mid \xi_0) &= \left(\sum_{i=1}^{m} \gamma_i(x)\,\alpha_i v_i \,\middle|\, \sum_{i=1}^{m} \alpha_i v_i \right) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m} \gamma_i(x)\alpha_i\bar{\alpha}_k(v_i \mid v_k) \\ &= \sum_{i=1}^{m} |\alpha_i|^2\,\gamma_i(x), \qquad x \in G, \end{aligned}$$

which yields (9).

We choose the matrix $[\beta_{i,j}] \in M(m \times n; \mathbb{C})$ in such a way that

$$v_i = \sum_{j=1}^{n} \beta_{i,j} e_j, \qquad i = 1, ..., m.$$

Then (8) results directly from the fact that both $(v_1, ..., v_m)$ and $(e_1, ..., e_n)$ are orthonormal systems of vectors from $\mathbb{C}^n$. Moreover,

$$\begin{aligned} a(x) = U(x)\xi_0 &= \sum_{i=1}^{m} \gamma_i(x)\,\alpha_i v_i \\ &= \sum_{i=1}^{m} \gamma_i(x)\,\alpha_i \sum_{j=1}^{n} \beta_{i,j}\, e_i \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_i \beta_{i,j} \gamma_i(x) \right) e_j, \qquad x \in G. \end{aligned}$$

Comparing this with (11), we conclude that (10) also holds.

To verify the converse statement of Theorem 2 one only needs to perform a simple calculation which we omit here.

3. A Functional Equation of D'Alembert's Type

At the beginning of the present section, $(G, \cdot)$ stands for an arbitrary group. Classical D'Alembert's equation for a complex-valued function f defined on G has the form

$$f(xy) + f(xy^{-1}) = 2f(x) f(y), \qquad x, y \in G. \tag{14}$$

The two most general results concerning Eq. (14) have been proved by Pl. Kannappan [3] and R. Dacić [1]. The first one, due to Kannappan, asserts that for any solution f of Eq. (14) satisfying the condition

$$f(xyz) = f(xzy), \qquad x, y, z \in G, \tag{15}$$

there exists a homomorphism m of the group G into the multiplicative group of the complex field $\mathbb{C}$ such that

$$f(x) = \tfrac{1}{2}(m(x) + m(x^{-1})), \qquad x \in G.$$

Dacić weakened condition (15) postulating its validity for a single point $z := z_0$ only. This profit was attained, however, at the expense of the additional assumption that f vanishes at the point z_0. It was shown in [1] that under the new hypotheses the assertion of Kannappan's result remains valid.

We prove an analogue of Dacić's result for Eq. (3) which, in the multiplicative notation for the group operation in G, assumes the form

$$f(xy)+f(xy^{-1})=2(a(x) \mid a(y)), \qquad x, y \in G. \tag{16}$$

We start with the following

PROPOSITION. *If the functions $f\colon G \to \mathbb{C}$ and $a\colon G \to H$ satisfy Eq.* (16), *then*

(i) $f(x)=(a(x) \mid a(e)), \qquad x \in G;$

(ii) $f(x)=f(x^{-1}), \qquad x \in G;$

(iii) $f(x)=\overline{f(x)}, \qquad x \in G;$

(iv) $f(xy)=f(yx), \qquad x, y \in G.$

Proof. Assertion (i) results on substituting $y := e$ to (16) and dividing both its sides by 2.

Inserting $x := e$ to (16), we get

$$f(y)+f(y^{-1})=2(a(e) \mid a(y))=2\overline{(a(y) \mid a(e))}=2\overline{f(y)}, \qquad y \in G,$$

whence,

$$f(y)+f(y^{-1})=2\overline{f(y)}, \qquad y \in G. \tag{17}$$

The left-hand side of (17) remains unchanged if we replace y by y^{-1}, so its right-hand side is not affected by this replacement either. This means that (ii) holds.

Now applying (ii) to (17), we immediately arrive at (iii).

Finally, from (16), (ii), and (iii) we infer that

$$\begin{aligned} f(yx)+f(yx^{-1}) &= 2(a(y) \mid a(x)) = 2\overline{(a(x) \mid a(y))} \\ &= \overline{f(xy)+f(xy^{-1})} = f(xy)+f(xy^{-1}) \\ &= f(xy)+f(yx^{-1}), \qquad x, y \in G. \end{aligned}$$

Subtracing $f(yx^{-1})$ from the outermost terms of these identities, we obtain (iv), which completes the proof.

THEOREM 3. *Let $(G, \cdot)$ be a topological group and let $f\colon G \to \mathbb{C}$ and $a\colon G \to H$ be continuous functions. Moreover, suppose that there exists a $z_0 \in G$ such that*

$$a(z_0)=0 \tag{*}$$

and

$$a(xyz_0)=a(xz_0y), \qquad x, y \in G. \tag{**}$$

If f and a satisfy Eq. (16), then there exists a continuous unitary representation U of the group G in the space $H_0 := \mathrm{cl\,Lin}\, a(G)$ with the cyclic vector $\xi_0 := a(e)$ such that

$$(U(xy)\xi_0 \mid \xi_0) = (U(yx)\,\xi_0 \mid \xi_0), \qquad x, y \in G; \qquad (\alpha)$$

$$a(x) = \tfrac{1}{2}(U(x)\,\xi_0 + U(x^{-1})\,\xi_0), \qquad x \in G; \qquad (\beta)$$

$$f(x) = \tfrac{1}{2}(U(x)\,\xi_0 + U(x^{-1})\,\xi_0 \mid \xi_0), \qquad x \in G. \qquad (\gamma)$$

Conversely, if $f\colon G \to \mathbb{C}$ and $a\colon G \to H$ (possibly without properties (*) and (**)) are defined by formulae (γ) and (β), respectively, with some unitary representation U and a vector ξ_0 satisfying (α), then f and a are solutions of Eq. (16).

Proof. Assume that $f\colon G \to \mathbb{C}$ and $a\colon G \to H$ are continuous functions satisfying Eq. (16) together with conditions (*) and (**). By (i) and (**) we have

$$\begin{aligned} f(xyz_0) &= (a(xyz_0) \mid a(e)) = (a(xz_0\,y) \mid a(e)) \\ &= f(xz_0\,y), \qquad x, y \in G, \end{aligned}$$

which means that (**) extends to f, i.e.,

$$f(xyz_0) = f(xz_0\,y), \qquad x, y \in G. \qquad (18)$$

Hence and from (iv) we infer that

$$f(xyz_0) = f(xz_0\,y) = f((xz_0)y) = f(yxz_0), \qquad x, y \in G.$$

Consequently, we have

$$f(xyz_0) = f(yxz_0), \qquad x, y \in G. \qquad (19)$$

Equations (16) and (*) imply that

$$f(xz_0^2) + f(x) = 2(a(xz_0) \mid a(z_0)) = 0, \qquad x \in G,$$

whence,

$$f(xz_0^2) = -f(x), \qquad x \in G. \qquad (20)$$

Let us also observe that

$$f(xz_0^{-1}y^{-1}) = -f(xy^{-1}z_0), \qquad x, y \in G. \qquad (21)$$

Indeed, from (20) and (18) we deduce that

$$\begin{aligned} f(xz_0^{-1}y^{-1}) &= -f(xz_0^{-1}y^{-1}z_0^2) = -f(xz_0^{-1}z_0y^{-1}z_0) \\ &= -f(xy^{-1}z_0), \qquad x, y \in G. \end{aligned}$$

Now, applying successively (16), (18), and (20), we derive

$$\begin{aligned} 2(a(xz_0) \mid a(yz_0)) &= f(xz_0yz_0) + f(xz_0z_0^{-1}y^{-1}) \\ &= f(xz_0(yz_0)) + f(xy^{-1}) = f(xyz_0^2) + f(xy^{-1}) \\ &= -f(xy) + f(xy^{-1}), \qquad x, y \in G, \end{aligned}$$

which combined with (16) yields

$$f(xy^{-1}) = (a(x) \mid a(y)) + (a(xz_0) \mid a(yz_0)), \qquad x, y \in G. \tag{22}$$

Further, using twice Eq. (16) and referring first to (18) and then to (21), we obtain

$$\begin{aligned} 2(a(xz_0) \mid a(y)) &= f(xz_0y) + f(xz_0y^{-1}) \\ &= f(xyz_0) + f(xy^{-1}z_0), \qquad x, y \in G \end{aligned}$$

and

$$\begin{aligned} 2(a(x) \mid a(yz_0)) &= f(xyz_0) + f(xz_0^{-1}y^{-1}) \\ &= f(xyz_0) - f(xy^{-1}z_0), \qquad x, y \in G. \end{aligned}$$

The last two identities combined together imply that

$$f(xy^{-1}z_0) = (a(xz_0) \mid a(y)) - (a(x) \mid a(yz_0)), \qquad x, y \in G. \tag{23}$$

After these auxiliary calculations we define two continuous functions $g: G \to \mathbb{C}$ and $b: G \to H$ by

$$\begin{aligned} g(x) &:= f(x) + if(xz_0), \qquad x \in G; \\ b(x) &:= a(x) + ia(xz_0), \qquad x \in G. \end{aligned}$$

Directly from (iv), (19), and the definition of g it follows that

$$g(xy) = g(yx), \qquad x, y \in G. \tag{24}$$

We show that g and b satisfy the equation

$$g(y^{-1}x) = (b(x) \mid b(y)), \qquad x, y \in G.$$

In fact, by (22), (23), and (24) we have

$$\begin{aligned}(b(x) \mid b(y)) &= (a(x)+ia(xz_0) \mid a(y)+ia(yz_0)) \\ &= (a(x) \mid a(y)) + (a(xz_0) \mid a(yz_0)) \\ &\quad + i[(a(xz_0) \mid a(y)) - (a(x) \mid a(yz_0))] \\ &= f(xy^{-1}) + if(xy^{-1}z_0) = g(xy^{-1}) = g(y^{-1}x), \qquad x, y \in G.\end{aligned}$$

According to Theorem 1, there exists a continuous unitary representation U of the group G in the space $H_0 := \mathrm{cl}\,\mathrm{Lin}\, b(G)$ with the cyclic vector

$$\xi_0 := b(e) = a(e) + ia(z_0) = a(e)$$

such that

$$b(x) = U(x)\xi_0, \qquad x \in G; \tag{25}$$

$$g(x) = (U(x)\xi_0 \mid \xi_0), \qquad x \in G. \tag{26}$$

Now, in view of (24) and (26), it becomes apparent that (α) holds. Moreover, (16) implies that

$$\begin{aligned}&2\,\|a(x) - a(x^{-1})\|^2 \\ &\quad = 2(a(x) \mid a(x)) - 2(a(x) \mid a(x^{-1})) \\ &\qquad - 2(a(x^{-1}) \mid a(x)) + 2(a(x^{-1}) \mid a(x^{-1})) \\ &\quad = f(x^2) + f(e) - f(e) - f(x^2) - f(e) - f(x^{-2}) \\ &\qquad + f(x^{-2}) + f(e) = 0, \qquad x \in G,\end{aligned}$$

whence,

$$a(x^{-1}) = a(x), \qquad x \in G. \tag{27}$$

Using again (16) together with (18) and (20), we obtain

$$\begin{aligned}&2\,\|a(xz_0) + a(x^{-1}z_0)\|^2 \\ &\quad = 2(a(xz_0) \mid a(xz_0)) + 2(a(xz_0) \mid a(x^{-1}z_0)) \\ &\qquad + 2(a(x^{-1}z_0) \mid a(xz_0)) + 2(a(x^{-1}z_0) \mid a(x^{-1}z_0)) \\ &\quad = f(xz_0(xz_0)) + f(e) + f(xz_0(x^{-1}z_0)) + f(x^2) + f(x^{-1}z_0(xz_0)) \\ &\qquad + f(x^{-2}) + f(x^{-1}z_0(x^{-1}z_0)) + f(e) \\ &\quad = f(x^2z_0^2) + f(e) + f(z_0^2) + f(x^2) \\ &\qquad + f(z_0^2) + f(x^{-2}) + f(x^{-2}z_0^2) + f(e) \\ &\quad = [f(x^2z_0^2) + f(x^2)] + [f(x^{-2}z_0^2) + f(x^{-2})] \\ &\qquad + 2[f(z_0^2) + f(e)] = 0, \qquad x \in G,\end{aligned}$$

which means that

$$a(x^{-1}z_0) = -a(xz_0), \qquad x \in G. \tag{28}$$

Turning back to the definition of b and employing (27) and (28), we get

$$\begin{aligned}\tfrac{1}{2}[b(x)+b(x^{-1})]\\ &= \tfrac{1}{2}[a(x)+a(x^{-1})] + \tfrac{1}{2}i[a(xz_0)+a(x^{-1}z_0)]\\ &= a(x), \qquad x \in G,\end{aligned}$$

which jointly with (25) yields (β). Moreover, the last identity combined with the definition of b ensures that $\operatorname{Lin} b(G) = \operatorname{Lin} a(G)$ and, therefore, $H_0 = \operatorname{cl} \operatorname{Lin} a(G)$. Finally, by (i) and ($\beta$), we have

$$f(x) = (a(x) \mid a(e)) = \tfrac{1}{2}(U(x)\xi_0 + U(x^{-1})\xi_0 \mid \xi_0), \qquad x \in G,$$

which completes the proof of the firrst part of the theorem. The proof of the converse statement relies on a direct verification that Eq. (16) holds and is therefore omitted.

Repeating the argument used in the proof of Theorem 2, one can deduce from Theorem 3 the following

THEOREM 4. *Let $(G, +)$ be an Abelian topological group and let $f, a_j\colon G \to \mathbb{C}$ $(j = 1, ..., n)$ be continuous functions. Moreover, suppose that there exists a $z_0 \in G$ such that*

$$a_j(z_0) = 0 \qquad \text{for} \quad j = 1, ..., n.$$

If the functions f and a_j $(j = 1, ..., n)$ satisfy the equation

$$f(x+y) + f(x-y) = 2 \sum_{j=1}^{n} a_j(x)\overline{a_j(y)}, \qquad x, y \in G, \tag{29}$$

then there exist continuous characters $\gamma_1, ..., \gamma_m$ of the group G $(m \leqslant n)$, complex numbers $\alpha_1, ..., \alpha_m$ and a matrix $[\beta_{i,j}] \in M(m \times n; \mathbb{C})$ such that

$$\sum_{j=1}^{n} \beta_{i,j}\bar{\beta}_{k,j} = \delta_k^i, \qquad i, k = 1, ..., m; \tag{30}$$

$$f(x) = \sum_{i=1}^{m} |\alpha_i|^2 \operatorname{Re} \gamma_i(x), \qquad x \in G; \tag{31}$$

$$a_j(x) = \sum_{i=1}^{m} \alpha_i \beta_{i,j} \operatorname{Re} \gamma_i(x), \qquad x \in G, j = 1, ..., n. \tag{32}$$

Conversely, arbitrary functions f and a_j, $j = 1, ..., n$ (possibly with no zeros), defined by (31) *and* (32), *with the elements $\beta_{i,j}$ satisfying* (30), *are solutions of Eq.* (29).

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